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Vortex structures generated by a coastal current in harbour-like basins at large Reynolds numbers

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It has been observed for a long time that under certain conditions a vortex or even a group of vortices forms in bays which have a narrow opening to the sea. What leads to the formation of such vortices confined in a quiet, almost closed bay? Why does their number vary? Can such vortices form in any specific bay with known hydrological conditions, coastal configuration and bottom topography? The answers to these questions are essential in practice because, if several vortices form in a bay, a sort of a 'vortex cork' is created which prevents the outflow of pollution from the bay. This pollution will be locked in the bay practically permanently. The formation of vortices can also very strongly modify the topology of the background flow and lead to the formation of structures which intensify such processes as beach drifting, silting, and coastal erosion.

This article considers the topology of the vortex regimes generated in harbour-like basins by the external potential longshore current at large Reynolds numbers. The theory discusses the issues of what solution compatible with the Prandtl–Batchelor theorem for inviscid fluids, and under what conditions, may be realized as an asymptotic state of the open hydrodynamical system. The analysis is developed based on the variational principle, the most appropriate fundamental method of modern physics in this case, modified for the open degenerated hydrodynamical system. It is shown that the steady state corresponds to the circulational regime in which the system has minimal energy and enstrophy. This state is fixed by the Reynolds number. The relation between the Reynolds number, the geometry factor and the topological number, characterizing the number of vortex cells, is found.

1. Introduction

It has been observed for a long time that under certain conditions a vortex or even a group of vortices forms in bays which have a narrow opening to the sea. What leads to the formation of such vortices confined in a quiet, closed bay? Why does their number vary?

Our study addresses these questions and presents a theory of the development and the maintenance of such complex multiple-vortex structures.

Despite a series of significant successes in studying and classifying vortex regimes within the conservative models of hydrodynamics, the question of *how* these vortices are realized in natural, non-conservative flows remains unanswered. The study of high-Reynolds-number flows around obstacles appears to be one of the most complicated

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problems of hydrodynamics (Lavrent'ev & Shabat 1973; Saffman 1981). Saffman's (1981) suggestion to approach the task by attempting to determine the asymptotic stationary solutions of the Navier–Stokes equations at zero viscosity did not turn out to be fruitful because of the degeneration of this system of equations. Even if it were possible to find all circulation regimes which can play the role of the asymptotic states, such an approach would not make it possible to classify the states in accordance with the Reynolds numbers. Even attempts to solve this problem with the use of numerical models encountered significant difficulties because of the non-trivial hierarchy of the circulation regimes in the domain of model parameters.

According to observations (Lavrent'ev & Shabat 1973), the vortices are arranged within the basin in such a manner that their signs alter. The theoretical explanation and the experimental verification of such a flow arrangement were given by Weiss & Florsheim (1965), Pan & Acrivos (1967).

The simple linear models in rectangular cavities agreed well with the observations, but only at relatively small Reynolds numbers. The reason why a similar theory has failed to be developed for large Reynolds numbers is that the application of the boundary layer theory for multi-cell problems appears to be complex and unwieldy. At the present time, this approach offers no way of determining the topology of the asymptotic state, but rather merely presents a procedure for determining the state parameters by joining together the inviscid flow and the flow in cyclic boundary layers. However, even so, such an approach had been employed for calculating the circulation flow with comparatively simple topology (see e.g. Mills 1965; Chernyshenko 1988; Chernyshenko & Castro 1993).

Recently, a new theoretical approach has been used by Goncharov & Makarov 1996. The basis of this approach is that under the influence of stationary external factors sustaining a weakly viscous flow, a hydrodynamic system reaches its steady state at a fixed enstrophy. By idealizing such a regime as a two-dimensional inviscid vortex, one can seek the solution from the variational principle because the states in which the hydrodynamic system has minimum energy with fixed enstrophy are determined by the Reynolds number.

The advantage of this approach is that it is not limited to the small-Reynolds-number range. In this article we employ this method to consider problems with very large Reynolds numbers $Re = LV/\nu$. We suppose that the fluid motions are characterized by the following typical scales: the horizontal, V, and vertical, W, velocities, as well as the horizontal length scale L. The layer thickness, d, will be regarded as the vertical length-scale. We assume that the Froude number, $Fr = V^2/gL$, is rather small, and that the Reynolds number, $Re_d = Vd/\nu$, in contrast, is large. The fluid layer is assumed to be rather thick, and motions quasi-two-dimensional, so that inequalities $V^2/gL \ll 1$, $\nu/Vd \ll 1$, $d/L \ll 1$, $W/V \ll 1$ hold true (g is the acceleration due to gravity, ν is the kinematic viscosity of the fluid). Such problems are typical for geophysical hydrodynamics. For example, if the size of a bay or a lagoon is on a scale of $l \sim 10^2$–10^3 m, then for a moderate velocity of flows $V \sim 1$–10^{-1} m s^{-1} and typical viscosity $\nu \sim 10^{-6}$ m^2 s^{-1}, the characteristic Reynolds number is $Re \sim 10^8 \gg 1$. Unfortunately, it is currently impossible to realize flow regimes with such values of the parameter Re in laboratory conditions. Thus, no available laboratory data exist to be compared with the estimates provided by our study. Our results, therefore, may serve as a theoretical guideline for numerical modelling, on the one hand, and qualitative experimental estimates, on the other. It should be kept in mind, though, that brute-force computer calculations of vortex structures are not very appealing because the results often depend on a number of factors of secondary importance, such as the coastline shape, which distort the overall picture by introducing details frequently even non-existent.

Instead of massaging the computer-calculated details of vortex structures, it is more important to answer the practical question of whether any vortices may form in any specific bay under certain external conditions. And if so, then how many vortices form under the given Reynolds number? What are the key geometrical parameters and how does the number of vortices depend on them? The answers to these questions are essential because if several vortices form in a bay, it creates a sort of a 'vortex cork' that prevents the outflow of pollution from the bay. This pollution will be locked in the bay practically permanently.

In addition, the formation of vortices can very strongly modify the topology of the background flow and lead to the formation of structures that intensify such processes as beach drifting, silting, and coastal erosion. The considerable role that the large-Reynolds-number circulation regimes play in transporting the sedimented matter in the coastal zone has already been realized.

We can now formulate one part of the problem in the following manner: What vortex structures are typical for a given coastal configuration, bottom topography, and hydrological conditions? How are they classified in terms of basic geophysical flow parameters such as the external Reynolds number Re, and geometrical factors ε?

Note here that there exists a second, more important, aspect of the problem – its role in the framework of fundamental physics. Is it possible to determine the number of vortices forming based on the variational principle – the most acceptable method of modern physics for this sort of problems? How should this principle be reformulated for an open hydrodynamical system (in the presence of a specified external flow) when the system under consideration is degenerated, and, consequently, when its final state is not determined by the system's minimal energy alone?

Our approach does not consider the problem of evolution under the influence of exterior causes from some initial state to a stationary one. (In fact, under such a consideration the energy would not remain unchanged.) We consider a completely different situation in which a weakly viscous system with $Re \gg 1$ has almost reached an equilibrium state. Such a situation has a relevance to flows found in nature. The Gulf Stream and the Big Red Spot on Jupiter are famous examples of quasi-steady natural quasi-two-dimensional flows with a very high Reynolds number and intense small-scale turbulent motions. It is known that the presence of turbulence does not destroy them. Let us note that even if a flow is turbulently unsteady, this does not necessarily mean that the average ordered motion is absent. In such a case, the laminar two-dimensional motion equations (see below, (14) and (15)) with effective (eddy) viscosity are known to be a good approximation of the description of large-scale average fields of a turbulent flow.

We show that if such a system evolves in a neighbourhood of the equilibrium state where its energy becomes constant, then a relaxation of enstrophy to a fixed value under the action of an exterior stationary cause sustaining the flow necessarily follows. This is the main feature of our approach.

As an illustration, in the sections devoted to dissipative systems evolving as nearly inviscid in the vicinity of an equilibrium state, we present both mechanical and field dynamical examples of this type of systems.

We consider the simple case of a vortex cell generated by an exterior stream flowing near a rectangular cavity. On the one hand, the cyclic boundary layer formed around the vortex cell is sustained by the exterior mechanism (pressure gradient, in our case), while on the other hand, it serves as a vorticity source. In accordance with the motion equations of viscous fluid, there exists a process of generating a vorticity flux across streamlines into the cell and outwards through the free boundary. As shown by Batchelor (1956), for the internal region where the influence of viscous forces is small compared with inertial forces, such a process ceases once a constant vorticity is reached. The availability of a flux through the free surface alters neither the qualitative, nor the quantitative analysis. This is not surprising because there exists a natural mechanism compensating for the losses – viscous dissipation.

2. Formulation of the problem and the variational principle for stationary regimes

2.1. *Preliminary remarks*

A closed system, left to itself, under the influence of dissipative effects comes to a state of minimal energy. However, if the system is open and dissipation is compensated by external dynamical processes, this compensation provides additional integrals of motion. Acting as constraints, these integrals cause the system to come to the state now determined not by the minimum of energy, but by its linear combinations with these integrals.

As an illustration, let us consider a simple mechanical system: a small ball with mass m moving in a viscous medium, along the surface of a unit radius sphere, in the gravity field g, and a force with the constant azimuth momentum K which compensates the viscosity action. (The hydrodynamical consideration of fluid motions in the framework of the Hamiltonian approach is given in §2.2.)

This system is attracted to a stationary regime which consists of the ball's rotation around the vertical axis with a constant angular velocity. In terms of the polar and azimuth angles, θ and φ, this state is determined by conditions

$$p_\theta = 0, \quad p_\varphi = K/v, \quad \frac{p_\varphi^2 \cos\theta}{ml^2 \sin^3\theta} - mgl \sin\theta = 0, \tag{1}$$

where $p_\theta = ml^2\dot{\theta}$ and $p_\varphi = ml^2 \sin^2\theta\dot{\varphi}$ are polar and azimuth angular momentums, respectively.

On the other hand, the same result may be obtained from the variational principle

$$\delta \left(H + \lambda p_\varphi\right) = 0, \tag{2}$$

where the Hamiltonian

$$H = \frac{1}{2ml^2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right) - mgl\cos\theta \tag{3}$$

and the azimuth momentum p_φ are the motion integrals of the inviscid problem the dynamics of which is described in terms of two pairs of canonically conjugated variables u_k : (p_θ, θ), (p_φ, φ). After varying the expression $H + \lambda p_\varphi$ with respect to variables p_θ, p_φ, θ, φ, i.e. calculating $\partial/\partial u_i\,(H + \lambda p_\varphi) = 0$, and setting $\lambda ml^2 \sin^2\theta = -K/v$, we arrive at (1), from which follows the equation

$$K^2 \cos\theta = gv^2m^2l^3 \sin^4\theta$$

for angle θ at which the system attains this regime.

When $K = 0$, the solution of the equation is $\theta = \pi$, and the ball will fall eventually to the bottom of the sphere to its state of minimum energy. If K is not equal to zero, the ball will be repeating steady circles with a fixed value of θ. However, if one attempts to formally tend ν and K to zero in order to make a transition to an ideal system, one will face the 0/0-type uncertainty in determining the final state of the system. This illustrates a typical scenario for ideal systems – the ball circulating at any constant value of θ.

2.2. *Hamiltonian approach*

It is well known that conservative systems evolve into a phase space along the surfaces determined by the motion integrals inherent in the given system. Generally speaking, the introduction of dissipation and external sources into the system radically changes the character of the evolution. However, in some cases the joint action of these factors can become responsible for the emergence of an effectively inviscid regime of motion.

For definiteness, let us consider the continuous-field Hamiltonian system whose dynamics are described by the equation

$$\partial_t u = \hat{\mathrm{J}} \frac{\delta H}{\delta u}, \tag{4}$$

where $u(\boldsymbol{x}, t)$ is the set of dynamical field variables, functional H under functional derivatives $(\delta/\delta u)$ is the Hamiltonian, and $\hat{\mathrm{J}}$ is the skew-symmetric Hamiltonian operator, satisfying the Jacobi condition.

For further mathematical discussion refer to Olver (1986), Salmon (1988), Goncharov & Pavlov (1993, 1997).

As a rule, in addition to the evident motion integral H, the system can also possess other motion integrals, I_k. The first group of them is associated with the symmetries of the Hamiltonian H for example the momentum which is related by Noether's theorem to the translational symmetry. The second group, the so-called Casimir invariants, owes its existence to the degeneracy of the operator $\hat{\mathrm{J}}$ and represents the annihilators of

$$\hat{\mathrm{J}} \frac{\delta I_k}{\delta u} = 0. \tag{5}$$

Thus, when seeking the stationary solution in a non-dissipative case (see, for example, Arnold 1969; Goncharov & Pavlov 1993 and references cited therein), from (4) and (5) we arrive at

$$\frac{\delta}{\delta u} \left(H + \sum_{k=1} \lambda_k I_k \right) = 0, \tag{6}$$

i.e. at the variational principle

$$\delta \left(H + \sum_{k=1} \lambda_k I_k \right) = 0, \tag{7}$$

where the arbitrary constants, λ_k, are the Lagrangian multipliers.

In the presence of dissipation and action of extraneous forces (sources), motion equations (4) can be modified to the form

$$\partial_t u = \hat{\mathrm{J}} \frac{\delta H}{\delta u} + s - \nu \varphi. \tag{8}$$

Here $s\,[u;\boldsymbol{x}]$ and $\varphi\,[u;\boldsymbol{x}]$ are the functions characterizing the action of sources and dissipative losses, respectively, and ν is the viscosity coefficient.

In this case the corresponding equations for the motion integrals of the inviscid problem take the form

$$\partial_t I_k = S_k - \nu \Phi_k, \tag{9}$$

where the functionals S_k and Φ_k characterizing the action of external sources and dissipative losses, respectively, are defined by the relationships

$$S_k = \int \frac{\delta I_k}{\delta u}\, s\,\mathrm{d}\boldsymbol{x}, \quad \Phi_k = \int \frac{\delta I_k}{\delta u}\, \varphi\,\mathrm{d}\boldsymbol{x}.$$

Unfortunately, it has not as yet been clearly identified what conditions should be imposed on the choice of the dissipative terms $s\,[u;\boldsymbol{x}]$ and $\varphi\,[u;\boldsymbol{x}]$ in order to be able to treat the steady solution of Hamiltonian system (7) as an attractor for dissipative modification (8). However, at least the very existence of such a possibility can be demonstrated with some examples.

One of the ways of constructing the dissipative modification for any Hamiltonian dynamical system was pointed out by Shepherd (1990) and consisted of the replacement of system (4) with the new system defined as

$$\partial_t u = \hat{\mathrm{J}}\frac{\delta H}{\delta u} + \hat{\mathrm{J}}\,\hat{\mathrm{A}}\,\hat{\mathrm{J}}\,\frac{\delta H}{\delta u} \tag{10}$$

where $\hat{\mathrm{A}}$ is a symmetric operator such that integral

$$A = \int u\,\hat{\mathrm{A}}\,u\,\mathrm{d}\boldsymbol{x} \tag{11}$$

has a definite sign for all u. Analogous modifications for various inviscid fluids have been used by Vallis, Carnevale & Young (1989) and Carnevale & Vallis (1990) for finding nonlinearly stable steady solutions that represent extrema of energy subject to given Casimir invariants characterizing the vortex topology.

Just like the original system, (4), the modified dynamical equation (10) preserves the same Casimir functionals and has the same steady solutions. However, now the Hamiltonian is no longer invariant, and since its derivative with respect to time has a definite sign, the Hamiltonian will monotonically increase or decrease according to this sign of $\hat{\mathrm{A}}$:

$$\partial_t H = \int \left(\hat{\mathrm{J}}\frac{\delta H}{\delta u}\right)\left(\hat{\mathrm{A}}\hat{\mathrm{J}}\frac{\delta H}{\delta u}\right)\mathrm{d}\boldsymbol{x}. \tag{12}$$

If the Hamiltonian, subject to the constraints imposed by the Casimir invariants, has a finite upper or lower bound, this process will stop at such a bound.

The system attains its steady regime, where

$$\hat{\mathrm{J}}\,\frac{\delta H}{\delta u} = 0. \tag{13}$$

Thus, the steady solutions following from variational principle (7) for the inviscid problem serve as attractors for the viscous problem.

Of course, in reality the mathematical structure of terms describing dissipative losses and external forcing in open real dynamical systems looks nothing like the one on the right-hand side of (10). Nevertheless, asymptotically in the neighbourhood of

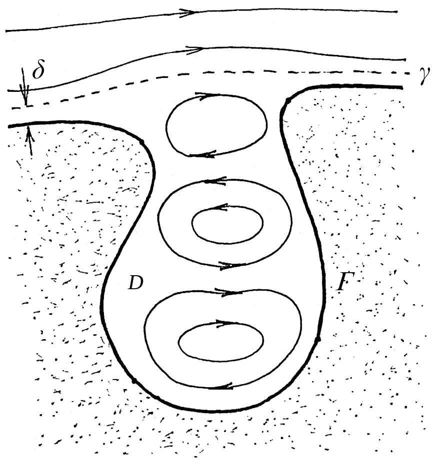

FIGURE 1. Sketch illustrating the streamline pattern of a multi-cell vortex structure generated by a potential longshore current in a harbour-like basin.

the attractor points, the phase trajectories of systems (8) may behave just like the trajectories of systems (10).

There is good reason to believe that such regimes can be realized in systems (8) under the influence of stationary sources $S_k = const$ when one of the dissipative functionals Φ_k on the right-hand side of (9) proves to be some function of the motion integrals H, I_k. This speculation is strengthened by the fact that, while reaching these regimes, the dissipative systems not only preserve the motion integrals, just like the conservative systems, but fix them, too.

2.3. *Basic equations for fluid motions: a variational problem*

In order to explore recirculational regimes excited by an external coastal flow in shallow basins with a horizontal scale greater than its depth, we will proceed from the system of equations of quasi-two-dimensional hydrodynamics

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla \left(\frac{p}{\rho}\right) + \nu \nabla^2 \boldsymbol{v} - \lambda_R \boldsymbol{v}, \tag{14}$$

$$\nabla \cdot \boldsymbol{v} = 0. \tag{15}$$

Here $\boldsymbol{v}(x, y)$ is the two-dimensional velocity field on the free surface, p is the pressure, ρ is the density, ν is the kinematic viscosity, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplacian operator, $\lambda_R = 2\nu/d^2$ is the bottom friction factor, d is the depth of the basin. The system of equations (14) and (15) is justified when describing rather slow motions in thin fluid layers (Gledzer, Dolzhanskii & Obuhov 1981).

A strict derivation of (14) and (15) can be made from the dynamic equations of viscous incompressible homogeneous fluid, if vertical motions and the curvature of the free surface may be ignored. In the shallow water approximation (when the characteristic horizontal length scale is much greater than the depth) variable $\mathbf{v}$ in (14) and (15) describes the hydrodynamic velocity on the free surface (see Appendix A).

We will consider (figure 1) that domain D occupied by a circulation flow is bounded on the coastal side by curve Γ, and on the ocean side by curve γ, which separates domain D from a longshore current, defined as the steady potential flow with the given velocity u at infinity. In this case, assuming that the vorticity $\omega = [\nabla, \boldsymbol{v}]$ is wholly concentrated inside domain D, for energy E,

$$E = \frac{1}{2}\int_D \boldsymbol{v}^2 \mathrm{d}\boldsymbol{x}, \tag{16}$$

contained in area D we can obtain from (14) and (15)

$$\partial_t E = \oint_{\Gamma+\gamma} \left(\left(\frac{\boldsymbol{v}^2}{2} + \frac{p}{\rho} \right) \boldsymbol{v} + \nu [\boldsymbol{\omega}, \boldsymbol{v}] \right) \cdot \mathrm{d}\boldsymbol{s} - \nu I - 2\lambda_R E, \tag{17}$$

where $\mathrm{d}\boldsymbol{s} = \mathrm{d}s\,\boldsymbol{n}$, $\boldsymbol{n}$ is the external normal vector to the contour and

$$I = \int_D \omega^2 \,\mathrm{d}\boldsymbol{x}. \tag{18}$$

On the right-hand side of (17), the first integral taken on the closed contour of domain D is simply the change of energy E due to its flux through this contour. The second and the third terms determine the decrease of energy E due to the internal and external friction, respectively.

Now, suppose that when $\nu \neq 0$ the energy flow is considered to be quasi-conservative, i.e. $\partial_t E = 0$. Obviously, such a case may be realized only if the balance

$$J = I + \frac{4}{d^2}E = \frac{1}{\nu}\oint_{\Gamma+\gamma} \left(\left(\frac{\boldsymbol{v}^2}{2} + \frac{p}{\rho} \right) \boldsymbol{v} + \nu [\boldsymbol{\omega}, \boldsymbol{v}] \right) \cdot \mathrm{d}\boldsymbol{s} \tag{19}$$

holds. In other words, this regime is realized if the energy flux into domain D exactly compensates viscous losses. It should be noted that flows of this type can be supported by such external factors as, for example, pressure gradient (see §5).

The study of quasi-conservative regimes, when external factors may be considered practically stationary, is a question of special interest. This scenario means that the vortex motion in domain D is organized in such a way that quantity J must remain fixed, i.e. be conserved. In this case, when the two-dimensional fluid motion is characterized by very large Reynolds numbers, the integral I (enstrophy) can be considered as the motion integral, as well as E.

Thus, flow regimes excited within the framework of the quasi-conservative scenario can be idealized as two-dimensional inviscid vortex flows in a bounded domain with additional motion integral I. Moreover, if we are interested in stationary solutions and follow the fundamental physical principle, which asserts that spontaneous change of a state of any open system occurs towards equilibrium, we come to the variational principle

$$\delta(E + \lambda I) = 0, \tag{20}$$

where λ is the Lagrangian multiplier. Equality (20) means that the equilibrium is reached in the points of phase space where functional E has a conditional extremum at fixed integral I.

The expression presented for the variational principle deserves further comments because the inviscid interpretation of this principle (Holm *et al.* 1985; Goncharov & Pavlov 1993) qualitatively differs from the viscous interpretation. Let us trace the reasons for this distinction by considering regimes with piecewise-constant field of vorticity

$$\omega = \sum_{k=1}^{N} \omega_k \theta_k, \tag{21}$$

where ω_k is the vorticity, θ_k is the characteristic theta-function ($\theta_k = 1$ if $\boldsymbol{x}_k \in s_k$, and $\theta_k = 0$ if $\boldsymbol{x}_k \notin s_k$), index k enumerates vortex cells ($k = 1, 2, ..., N$), N is the number of cells. This choice of vorticity is not accidental. Such states are asymptotically compatible with the Prandtl–Batchelor theorem (Prandtl 1905; Batchelor 1956). According to this theorem, if none of the closed streamlines of the flow pass through the boundary layers where viscous forces are comparable with inertial ones, then for $\nu \to 0$ there will exist a unique state of the flow characterized by constant vorticity.

The inviscid version of variational principle (20) for states (21) permits variations only with respect to the dynamic variables in this case. However, quantities ω_1, ω_2, ..., ω_N, which serve as the motion constants, may be set arbitrarily. As a result, in the inviscid approach there exists an uncertainty regarding which state, compatible with the Prandtl–Batchelor theorem, should be considered as asymptotic.

The concept of small viscosity allows us to resolve such an uncertainty. As shown (Petviashvili, Pokhotelov & Chudin 1982), the flow regimes for which the ideal fluid Hamiltonian reaches its extremum in the phase space may serve as attractors of real hydrodynamic systems with non-zero viscosity for two reasons. First, even though these regimes are unstable, the system spends a much longer time in the neighbourhoods of the attractors than away from them. Second, the existence of viscosity lifts the restriction typical for ideal fluid that is caused by the existence of an infinite number of integrals of motion and reflects the fact that a regime of interest may be unattainable from arbitrary initial conditions. These reasons explain the tendency of weakly viscous fluids to form coherent vortex structures: long-living, large-scale vortex objects. This effect is observed both numerically (Montgomery *et al.* 1992; Biskamp & Welter 1990; Dritschel 1993) and by laboratory experiments (Couder & Basdevant 1986; Nguyen Duc & Sommeria 1988) which show that after an initial transient period, a vorticity field evolves into an accidental ensemble of quasi-steady and localized vortices which can be interpreted as the most probable equilibrium states of inviscid fluid (see e.g. Pasmanter 1994; Robert & Sommeria 1991; Shen 1992 and references cited there).

In the viscous version of the variational principle which we consider in our study, viscosity plays a role similar to that in real hydrodynamic systems. In the presence of weak viscosity, quantities $\omega_1, \omega_2, ..., \omega_N$ are no longer motion integrals. Thus, the flow is permitted to evolve to the most probable state (described by such a set of $\omega_1, \omega_2, ..., \omega_N$) at which the system has its minimum energy for the fixed value of I

$$I = \sum_{k=1}^{N} \omega_k^2 s_k, \tag{22}$$

$$s_k = \int \theta_k \mathrm{d}\boldsymbol{x}, \tag{23}$$

where s_k is the area of the kth cell. This implies that in the approximation of small viscosity, the corresponding variational principle (20) permits variations of all variables capable of evolving in the phase space. Among these variables are the *unfrozen* variables ω_1, ω_2, ... ω_N. Thus, variational principle (20) is qualitatively different from its counterpart for ideal fluid because it is formulated for the most probable steady states.

To avoid consideration of local effects, we will formulate the theory in terms of effective cell-average parameters. The reason for doing this is that approximation (21) is rather rough and should be considered as nothing more than a qualitative

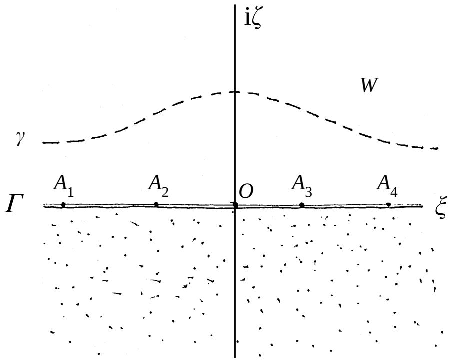

FIGURE 2. Sketch illustrating the transformation $z = z(W)$.

description of the global structure of the vortex field. It is well known that no matter how small the viscosity is, such solutions become locally unsuitable within the neighbourhood of streamlines identified with the contours of cells and within the streamlined relief where viscous effects cannot be neglected. In reality, the cyclic boundary layers develop in place of the separatrix streamlines surrounding the vortex cells. In principle, the asymptotic state may be fixed as a result of matching the inviscid flow with the flow in the boundary layer. However, such an approach is not trivial and so far has been realized only for circulation flows with a comparatively simple topology.

3. Flow around a polygonal shoreline in the presence of finite-area regions of constant vorticity

For the following analysis, we use the methods of the theory of complex variables. Let us assume that there exists transformation $W = W(z)$ ($z = x + \mathrm{i}y$, $W = \xi + \mathrm{i}\zeta$, i is the imaginary unit), which, in the vicinity of an infinitely remote point, has the form $W = cz$ and conformally maps domain D onto the upper half-plane $\zeta = \mathrm{Im}W \geqslant 0$, so that curve-linear boundary Γ, simulating the shoreline, is transformed into a straight line $\zeta = 0$ (figure 2). If such a transformation is known, the full stream function Ψ of an inviscid two-dimensional flow may be determined according to the theory of complex variables (see e.g. Lavrent'ev & Shabat 1965).

The boundary-value problem can be formulated in terms of Ψ and ω as follows:

$$\nabla^2 \Psi = -\omega, \tag{24}$$

where the continuous function Ψ has continuous partial derivatives and satisfies the following boundary conditions:

$$\left.\begin{aligned} \Psi &= 0 \text{ on } z \in \Gamma, \\ \partial\Psi/\partial y &= u \text{ as } x \to \pm\infty. \end{aligned}\right\} \tag{25}$$

By virtue of the fact that solution ψ_p of the homogeneous problem for $\omega = 0$ (when only the potential flow takes place) and the Green function, G, for problem (24) and (25) are well known in terms of W,

$$\psi_p = \frac{u}{c}\mathrm{Im}W, \quad G(z, z') = -\frac{1}{2\pi}\ln\left|\frac{W(z) - W(z')}{W(z) - \overline{W(z')}}\right|,$$

the total solution is described by Green's formula

$$\Psi = \psi_p + \psi_v \equiv \frac{u}{c}\mathrm{Im}W + \frac{1}{2\pi}\int \omega' \ln\left|\frac{W - W'}{W - \overline{W'}}\right| \mathrm{d}\boldsymbol{x}'. \tag{26}$$

Here, terms ψ_p and ψ_v describe the potential and vortex portions of the flow, respectively. Also, henceforth the primed field variables denote dependence on primed coordinates; the overline is the symbol for complex conjugation. This result allows us to express energy functional (16) in terms of vorticity ω. Since functional E in variational principle (20) is defined with accuracy up to a constant independent of the dynamical variables, in calculating E we may omit the ω-independent portion of the kinetic energy associated with the purely potential flow. Using this fact, we obtain in terms of stream functions

$$E = \frac{1}{2}\int \left[(\nabla\Psi)^2 - \left(\nabla\psi_p\right)^2\right] \mathrm{d}\boldsymbol{x} = \frac{1}{2}\int (2\psi_p + \psi_v)\omega \mathrm{d}\boldsymbol{x} - \frac{1}{2}\int (2\psi_p + \psi_v)\nabla\psi_v \cdot \mathrm{d}\boldsymbol{s},$$

Here, all integrals are taken on the whole domain of the flow with the exception of the last integral, which is taken along boundary Γ where $\psi_p = \psi_v = 0$. The last integral, therefore, becomes zero. Thus, E is given by

$$E = \frac{1}{2}\int (2\psi_p + \psi_v)\omega \mathrm{d}\boldsymbol{x}.$$

Substituting the stream functions from (26), this expression becomes

$$E = \frac{u}{c}\int \omega \mathrm{Im}W \mathrm{d}\mathbf{x} - \frac{1}{4\pi}\int \omega\omega' \ln\left|\frac{W - W'}{W - \overline{W'}}\right| \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}'. \tag{27}$$

Since there is little hope of resolving the problem with analytical methods in the frame of the continuous model, it is common practice to consider some ways of replacing it by a lower-order, discrete analogue. This approach is based on the use of different approximations for a continuous vorticity field that grasps the general structure of the field qualitatively correctly. In our case, such an approximation is (21) which is asymptotically compatible with the Prandtl–Batchelor theorem. The use of this approximation for functional E leads to the following estimate:

$$E = \frac{u}{c}\sum_{k=1}^{N} e_k\omega_k - \frac{1}{4\pi}\sum_{k\geqslant n=1}^{N} e_{k,n}\omega_k\omega_n, \tag{28}$$

$$e_k = \int \theta_k \mathrm{Im}W \mathrm{d}\boldsymbol{x}, \tag{29}$$

$$e_{k,n} = \left(2 - \delta_{k,n}\right)\int \theta_k\theta_n' \ln\left|\frac{W - W'}{W - \overline{W'}}\right| \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}'. \tag{30}$$

Here, N is the number of cells, δ_{kn} is the Kronecker delta-symbol, s_k is the area of the kth cell. On the right-hand side of (28), the first term is caused by the interaction of vortex cells with the stream, and the second term is determined by pairwise interactions of the cells with each other and takes into account the contributions of the cells' own energies as well.

In a general case it is impossible to identify the analytical expression of conformal mapping W for any contour Γ. Consequently, various sorts of approximations are commonly employed (Lavrent'ev & Shabat 1965, 1973). The essence of some of them consists of the replacement of the real boundary Γ by an open n-gon, which is characterized in the z-plane by vertices A_k and corners $\pi\alpha_k$ ($0 < \pi\alpha_k \leqslant 2, k = 1, 2, ..., n$) at the vertices. Then, according to the Schwartz–Christoffel theorem, the desired conformal mapping onto the half-space $\mathrm{Im}W \geqslant 0$ is given by

$$\frac{1}{c}\frac{\partial W}{\partial z} = \prod_{k=1}^{n} (W - a_k)^{1-\alpha_k}, \tag{31}$$

where a_k are the images of the vertices, i.e. $a_k = W(A_k)$ ($\alpha_1 + \alpha_2 + \cdots \alpha_n = n$).

In practice, the problem of the construction of a conformal mapping is reduced to searching for images of the vertices, a_k, the constant, c, and the constant of integration. Representation (31) has such a degree of arbitrariness that any three points of a_k can be chosen for the sake of convenience. Then other points and constants are determined unambiguously.

If boundary Γ has angular points protruding into the flow, an additional condition should be formulated as the natural physical requirement of the finiteness of the hydrodynamic velocity at these points. In our case this requirement, known as the Kutta–Zhukovskiy condition, has the form

$$\pi\frac{u}{c} + \sum_{k=1}^{N} \gamma_k \omega_k = 0, \; \gamma_k = \int \frac{\theta_k \mathrm{Im} W}{\left|a_m - \overline{W}\right|^2} \mathrm{d}\boldsymbol{x}, \tag{32}$$

where a_m are only the points for which $\alpha_m > 1$.

Integrals e_{kn}, e_k, γ_k and s_k are evaluated in Appendix B.

4. Model of a rectangular harbour

To illustrate the method, we now consider the simplest configuration – the model of a rectangular basin with width l and length h (see figure 3).

We define the symmetry of the model as follows:

$$\left.\begin{aligned} W(A_4) = -W(A_1) = a, \quad & W(A_3) = -W(A_2) = 1, \\ z(A_4) = -z(A_1) = l/2, \quad & z(A_3) = -z(A_2) = l/2 - \mathrm{i}h. \end{aligned}\right\} \tag{33}$$

Then, according to (31), the conformal mapping of the flow domain onto the upper half-plane $\mathrm{Im}W \geqslant 0$ has the form

$$z = \frac{1}{c}\int_0^W \left(a^2 - t^2\right)^{1/2} \left(1 - t^2\right)^{-1/2} \mathrm{d}t - \mathrm{i}h. \tag{34}$$

From (33), (34) it follows that the parameters a and c satisfy the system of equations

$$hc = a\left[\mathrm{K}\left(1 - a^{-2}\right)^{1/2} - \mathrm{E}\left(1 - a^{-2}\right)^{1/2}\right], \quad lc = 2a\mathrm{E}\left(a^{-1}\right), \tag{35}$$

where K and E are the complete elliptic integrals of the first and second kind, respectively.

In the approximation $h/l \geqslant 1$, to which we will limit our consideration, it follows from (35) to a sufficient degree of accuracy that

$$a = \tfrac{1}{4}\exp\left(1 + \pi\frac{h}{l}\right), \quad c = \frac{\pi}{4l}\exp\left(1 + \pi\frac{h}{l}\right). \tag{36}$$

Our model becomes simpler if the assumption is made that all vortex cells have an almost rectangular form. This reflects the experimentally observable fact (Weiss & Florsheim 1965; Pan & Acrivos 1967; Lavrent'ev & Shabat 1973) that separatrix streamlines dividing the flow in a rectangular cavity into vortex cells are with almost straight lines. Moving along them, fluid particles leave the lateral wall at right angles from the separation points with ordinates h_n and reach the other wall at attachment points with the same ordinates. We will assume here that h_n are arranged in an ordered fashion such that $h = h_N > h_{N-1} > \dots h_{n+1} > h_n > h_n > \dots h_1 > 1$, and also that inequality $(h_n - h_{n-1})/l \geqslant 1$ is valid. By virtue of a one-to-one correspondence between z and W, this is approximately equivalent to the statement that, in the W-plane, boundaries of the vortex cells represent the system of semicircles $\hat{W} = \rho_k e^{is}$ (s is the curve parameter, $0 \leqslant s \leqslant \pi$) with radii

$$a = \rho_N \gg \rho_{N-1} \gg \dots \gg \rho_n \gg \dots \rho_1 \gg 1. \tag{37}$$

Even for the outer cells that undergo the strongest influence of the external current, the calculation according to (34) shows that the curve describing the outer boundary, γ, deviates only slightly from the straight line passing through angular points A_1 and A_2. The maximal deviation, Δh, is estimated at $\sim 0.17l$.

Assuming, in accordance with inequality (37), that $\rho_n/\rho_k \approx \beta^{k-n}$ where $\beta \ll 1$ is a small parameter, we shall then restrict ourselves to consideration of the terms of the first order in β. As this analysis shows, the contribution of these terms to the energy is due to factors $e_{k,k}$, $e_{k,k-1}$, $e_{k,k-2}$, for which, in the leading-order approximation, we have the following expressions:

$$e_{k,k} \simeq -16\left(\frac{a}{c}\right)^4 \left(\ln\left(\frac{\rho_k}{\rho_{k-1}}\right) - (1-d_k) + \frac{\rho_{k-1}}{\rho_k}(1-b_k)\right), \tag{38}$$

$$e_{k,k-1} \simeq -16\left(\frac{a}{c}\right)^4 \left(1 - \frac{\rho_{k-1}}{\rho_k}(1-b_k) - \frac{\rho_{k-2}}{\rho_{k-1}}\right), \tag{39}$$

$$e_{k,k-2} \simeq -16\left(\frac{a}{c}\right)^4 \frac{\rho_{k-2}}{\rho_{k-1}}, \tag{40}$$

where $d_k = 0.31\delta_{N,k} + 0.15\delta_{1,k}$ and $b_k = 0.42\delta_{N,k}$ are the correction factors not vanishing only at $k = 1$ or $k = N$ and appearing due to the particular position of these cells in the basin. Note that in calculating the interaction factors for $k = N, 1$, it is necessary to set $\rho_0 = 1$, $\rho_N = a$ in formulas (38)–(40). Results (38)–(40) mean that in the first-order approximation, each cell interacts only with the two nearest cells, while the interaction with the others can be neglected.

As similar calculations of the interaction factors with the longshore current show, in the leading-order approximation, the current interacts only with the nearest two vortex cells N and $N-1$. The corresponding interaction factors are given by

$$e_N = 2\frac{a^2}{c^2}(C_e a - \rho_{N-1}), \quad e_{N-1} = 2\frac{a^2 \rho_{N-1}}{c^2}, \tag{41}$$

where $C_e = 1.16$ is the correction factor.

The same two cells make the main contribution to condition (32) defining the finiteness of the hydrodynamic velocity in angular points A_1 and A_4 of the harbour.

The corresponding factors are determined (in the leading order) as

$$\gamma_N = \frac{2a}{c^2}(C_\gamma - \frac{\rho_{N-1}}{a}), \quad \gamma_{N-1} = \frac{2\rho_{N-1}}{c^2}, \tag{42}$$

where $C_\gamma = 1.70$ is the correction factor.

An estimate of energy E can be obtained by selecting from sum (28) all the terms no smaller than $O(\beta)$. It is convenient to non-dimensionalize the problem by defining

$$\mu_k = \frac{2a}{\pi uc}\omega_k,$$

in which case the leading-order approximation of E can be written as

$$E = \pi\left(\frac{ua}{c}\right)^2\left\{\sum_{k=1}^{N}\left[\mu_k^2\left(\ln\left(\frac{\rho_k}{\rho_{k-1}}\right) - (1-d_k) + \frac{\rho_{k-1}}{\rho_k}(1-b_k)\right) + \mu_k\mu_{k-2}\frac{\rho_{k-2}}{\rho_{k-1}}\right.\right.$$
$$\left.\left. + \mu_k\mu_{k-1}\left(1 - \frac{\rho_{k-1}}{\rho_k}(1-b_k) - \frac{\rho_{k-2}}{\rho_{k-1}}\right)\right] + \mu_N C_e + (\mu_N - \mu_{N-1})\frac{\rho_N}{a} + O(\beta^2)\right\}. \tag{43}$$

In a similar fashion, we can find the estimate for the enstrophy integral, I. Using the results for the cell areas s_k, in accordance with (22) we find

$$I = \frac{\pi^3 u^2}{4}\sum_{k=1}^{N}\mu_k^2\left[\ln\left(\frac{\rho_k}{\rho_{k-1}}\right) + f_k\right] + O(\beta^2), \tag{44}$$

where $f_k = -0.31\delta_{N,k} + 0.69\delta_{1,k}$ is the correction factor different from zero only at $k = 1$ or $k = N$. In this case, as earlier, $\rho_0 = 1$, $\rho_N = a$.

Now, by following the standard procedure, we come to variational principle (20) for a discrete model based on estimations (43) and (44). By differentiating the quantity $E + \lambda I$ with respect to μ_k and ρ_k and after some algebraic manipulation, we obtain two sets of equations:

$$\left.\begin{aligned}
&2\mu_1[(1+\lambda)\ln\rho_1 - (1-d_1) + \lambda f_1] + \frac{2\mu_1 - \mu_2}{\rho_1} + \frac{\rho_1}{\rho_2}(\mu_3 - \mu_2) = 0,\\
&\qquad\vdots\\
&2\mu_k\left[(1+\lambda)\ln\left(\frac{\rho_k}{\rho_{k-1}}\right) - 1\right] + \frac{\rho_k}{\rho_{k-1}}(2\mu_k - \mu_{k-1} - \mu_{k+1})\\
&\qquad + \frac{\rho_{k-2}}{\rho_{k-1}}(\mu_{k-2} - \mu_{k-1}) + \frac{\rho_k}{\rho_{k+1}}(\mu_{k+2} - \mu_{k+1}) = 0,\\
&\qquad\vdots\\
&C_e + 2\mu_N\left[(1+\lambda)\ln\left(\frac{a}{\rho_{N-1}}\right) - (1-d_N) + \lambda f_N\right]\\
&\qquad + \frac{\rho_{N-1}}{a}[(1-b_N)(2\mu_N - \mu_{N-1}) - 1] + \frac{\rho_{N-2}}{\rho_{N-1}}(\mu_{N-2} - \mu_{N-1}) = 0,
\end{aligned}\right\} \tag{45}$$

and

$$\left.\begin{aligned}(1+\lambda)(\mu_1+\mu_2)+\frac{\rho_1}{\rho_2}(\mu_3-\mu_2)+\frac{1}{\rho_1}\mu_1=0,\\ \vdots\\ (1+\lambda)\frac{\mu_k+\mu_{k+1}}{\rho_k}+\frac{\mu_{k+2}-\mu_{k+1}}{\rho_{k+1}}\frac{\rho_{k-1}}{\rho_k^2}(\mu_{k-1}-\mu_k)=0,\\ \vdots\\ 1+(1+\lambda)\frac{a}{\rho_{N-1}}(2\mu_N-\mu_{N-1})-\mu_N(1-b_N)-a\frac{\rho_{N-2}}{\rho_{N-1}^2}(\mu_{N-1}-\mu_N)=0.\end{aligned}\right\} \quad (46)$$

In order for the set of equations (45) and (46) to be complete, we must add condition (32) which in the leading-order approximation has form

$$1+\mu_N C_\gamma+\frac{\rho_{N-1}}{a}(\mu_N-\mu_{N-1})=0. \quad (47)$$

5. Discussion and results

The numerical analysis of equations (45)–(47) at fixed $N \geqslant 1$ and parameters $h/l \geqslant N$ (i.e. when vortex cells are not too packed), permits us to make two important generalizations. First, the vorticities of all cells have approximately identical absolute values and alternate in signs $\mu_k = -\mu_{k+1}$. Second, all circulation cells have very similar vertical size h/N and, therefore, the ordinates of the cell interfaces can be described by the formula $h_k = hk/N$. The integration of (34) gives in the first-order approximation the relation $\rho_k = 0.5\exp(\pi h_k/l)$. Its above-discussed properties lead to

$$\rho_k = \tfrac{1}{2}\exp\left(\pi\frac{hk}{Nl}\right). \quad (48)$$

This means that radii ρ_k, which define the cell boundaries in W-space, comprise a geometric progression. The same fact can be directly seen from (46) and (47), if we use

$$\mu_k \simeq \mu_N(-1)^{N-k}. \quad (49)$$

It follows from (47) that dimensionless quantity μ_N characterizing the vorticity of Nth cell is constant $\big(\mu_N = 1/C_\gamma + O(\beta) \approx 0.58\big)$ and in practice does not depend on the input model parameters.

Now, taking into account that by definition small parameter β is equal to ρ_{k-1}/ρ_k, we find its concrete value

$$\beta = \exp\left(-\pi\frac{h}{Nl}\right).$$

Therefore, the smallness of β, and consequently, the validity of the results and conclusions of the theory, will be ensured if $h/Nl \sim 1$.

By substituting (48) and (49) into (43) and (44), and by neglecting terms of $O(\beta)$ and higher, we obtain first-order approximations to the integrals of energy and enstrophy:

$$E = \pi\left(\frac{ua}{c}\right)^2\mu_N^2\left[\pi\frac{h}{l}-2N-C\right]+O(\beta), \quad I = \left(\frac{\pi ua}{2c}\right)^2\mu_N^2 hl + O(\beta), \quad (50)$$

where $C = C_e C_\gamma + 2C_s - d_N - d_1 \simeq 0.9$ is a constant not dependent on the parameters of the model.

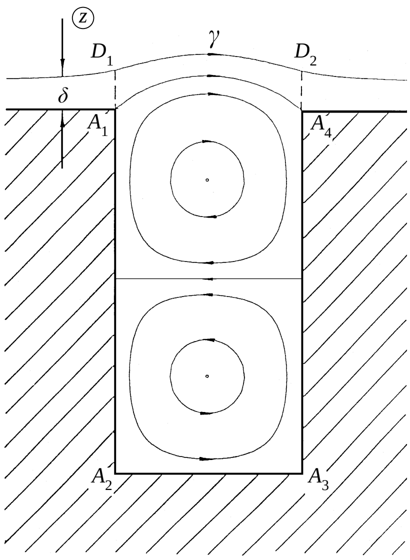

FIGURE 3. Definition sketch of a rectangular harbour model and schematic illustration of the vortex flow topology generated by a potential longshore current.

As shown in §2, if the far-flow velocity, u, and the geometry of the harbour are specified, multi-cell regimes of the circulation will be realized at a fixed value of quantity J. However, since the shallow water approximation used here implies $d \ll l$, it is possible to neglect term I in estimating J in formula (19). The expression for J is then

$$J = \frac{4}{d^2}E + O\left(\frac{d^2}{l^2}\right) \simeq 4\pi\left(\frac{ua}{dc}\right)^2 \mu_N^2 \left[\pi\frac{h}{l} - 2N - C\right]. \tag{51}$$

On the other hand, according to (19) quantity J is determined by contour integral

$$\frac{1}{\nu}\oint\left(\left(\frac{\boldsymbol{v}^2}{2} + \frac{p}{\rho}\right)\boldsymbol{v} + \nu[\boldsymbol{\omega}, \boldsymbol{v}]\right)\cdot \mathrm{d}\boldsymbol{s}$$

and can be fixed by the external pressure difference supporting the longshore current.

To estimate this effect, we will consider the energy flux through the controlling closed contour $D_1A_1A_2A_3A_4D_2$ (see figure 3), enclosing the circulation domain in such a manner that side D_1D_2 coincides with the streamline, γ, along the upper bound of the viscous boundary layer where $\boldsymbol{v}\cdot \mathrm{d}\boldsymbol{s} = 0$ and ω is small enough for the integral along D_1D_2 to be negligible. In such a situation, if D_1A_1 and D_2A_4 are chosen so that $[\boldsymbol{v}, \mathrm{d}\boldsymbol{s}] = 0$, it is obvious that only these two sides of the control contour contribute to the energy flux integral.

Besides, from symmetry considerations it is natural to assume that only pressure differs at these cross-sections. As result, we arrive at the estimation

$$J \simeq \frac{u\delta\Delta p}{2\nu\rho}. \tag{52}$$

Here δ is the characteristic thickness of the boundary layer, and Δp is the pressure difference across the harbour width l. If we introduce the resistance factor c_f determined by the ratio of frictional force $\delta\Delta p/l$ acting per unit of the surface to the

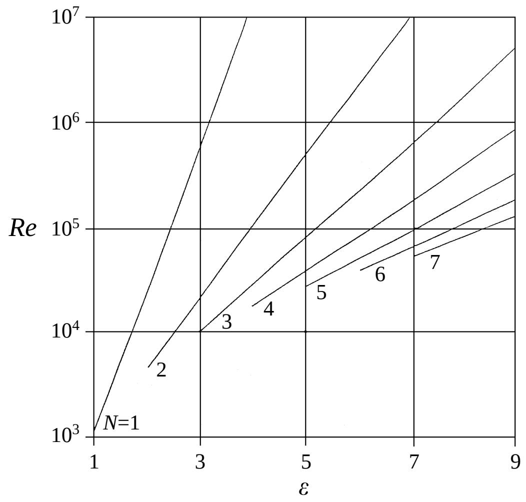

FIGURE 4. Classification of the topology (N) of multi-cell regimes in the plane of parameters Re and $\varepsilon = h/l$.

velocity head $\rho u^2/2$ (Schlichting 1955), estimation (52) can be rewritten in the more informative way

$$J = \frac{1}{4}\left(\frac{ul}{d}\right)^2 c_f Re, \tag{53}$$

where $Re = ud^2/\nu l$ is the Reynolds number determined in terms of the external friction. Such a representation of integral J is convenient for the introduction and use of various parameterizations for the resistance factor. The dependence of this factor on the Reynolds number is well known in a rather wide range of applications (Schlichting 1955). By comparing (51) and (53), we find condition

$$c_f Re = \frac{16\mu_N^2}{\pi}\left[\pi\frac{h}{l} - 2N - C\right] \tag{54}$$

under which the quasi-conservative vortex regime with N cells can be reached in the harbour. When constructing a diagram showing Re as a function of $\varepsilon = h/l$ for fixed values of N, we consider here for simplicity that resistance factor c_f does not depend substantially on the Reynolds number. The results of the calculations using (54) for the case $c_f = 0.001$ are depicted in figure 4. This figure shows that a plot of Re versus $\varepsilon = h/l$ behaves as a straight line with a constant inclination.

The results presented in figure 4 allow us to draw two main conclusions. First, the increase (decrease) of the number of vortex cells in a bounded harbour occurs due to the decrease (increase) of the Reynolds number and, correspondingly, of the external current velocity. Second, the narrower the harbour is, the more vortex cells are formed, but the less intensive they are. Even though the theory presented here says nothing about the intermediate regimes which are expected when the problem parameters do not satisfy the relation discovered for Re, several general considerations on this point can nevertheless be adduced. For example, in order to describe such regimes one can slightly modify the initial model by adding two angular vortices. Consideration of the angular vortices makes it possible to construct a physically realistic scenario of the development of intermediate regimes. The growth and the merger of angular vortices in such a model may be interpreted as the formation of the bottom cell and, by contrast, the dissipation and disintegration of them as the degeneration of the bottom cell.

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Appendix A. Bottom friction effect in a shallow-water model

Let us consider rather slow motions in an infinitely extended layer of incompressible viscous homogeneous fluid in a gravity field with the assumption of a rigid bottom and a free upper surface. The governing equations for this flow are the Navier–Stokes equations and the continuity equation supplemented by the boundary conditions, one of which implies the absence of viscous stresses at the free upper surface, and the other of which represents the no-slip condition at the bottom. Suppose that the fluid motions are characterized by the following typical scales: the horizontal, V, and vertical, W, velocities, as well as the horizontal length scale L. The layer thickness, d, will be regarded as the vertical length scale. Further, we assume that the Froude number, $Fr = V^2/gL$, is rather small, and that the Reynolds number, $Re_d = Vd/\nu$, in contrast, is large. The fluid layer is assumed to be rather thick, and motions quasi-two-dimensional, so that inequalities $V^2/gL \ll 1$, $\nu/Vd \ll 1$, $d/L \ll 1$, $W/V \ll 1$ hold (g is the gravity acceleration, ν is the kinematic viscosity of the fluid). Also, the free water surface is approximated as a frictionless rigid (horizontal) plate. In the hydrostatic approximation the motion equations and the boundary conditions are

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla + w\partial_z)\boldsymbol{v} = -\nabla\left(\frac{p}{\rho}\right) + \nu(\nabla^2 + \partial_z^2)\boldsymbol{v}, \quad g = -\partial_z \frac{p}{\rho}, \tag{A 1}$$

$$\nabla\cdot\boldsymbol{v} + \partial_z w = 0, \tag{A 2}$$

$$\partial_z \boldsymbol{v} = 0 \quad w = 0 \quad \text{at} \quad z = 0, \tag{A 3}$$

$$\boldsymbol{v}|_{z=d} = 0, \quad w|_{z=d} = 0 \quad \text{at} \quad z = d(x, y), \tag{A 4}$$

where $\boldsymbol{v}$ and w are the horizontal and vertical components of the velocity, $\nabla = (\partial_x, \partial_y)$ and ∂_z are the horizontal and vertical gradient operators, p is the pressure, ρ is the constant density, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional (horizontal) Laplacian operator, the vertical coordinate, z, is aligned opposite to the direction of gravity.

In the shallow water approximation $L \gg d$. Using the Taylor theorem, the field of the velocity can be expanded in the vicinity of the free upper surface $z = 0$ into a power series of the vertical coordinate z. By restricting ourselves to the first three terms, we obtain

$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_0' z + \tfrac{1}{2}\boldsymbol{v}_0'' z^2, \qquad w = w_0 + w_0' z + \tfrac{1}{2} w_0'' z^2, \tag{A 5}$$

where the continuity equation imposes the constraints on the expansion coefficients

$$w_0' = -\nabla\cdot\boldsymbol{v}_0, \qquad w_0'' = -\nabla\cdot\boldsymbol{v}_0'. \tag{A 6}$$

Now, if we substitute expansions (A 5) into boundary conditions (A 3) and (A 4), we find in view of relations (A 6) that

$$\nabla \cdot \boldsymbol{v}_0 = 0, \quad \boldsymbol{v}_0' = 0, \quad \boldsymbol{v}_0'' = -\frac{2}{d^2}\boldsymbol{v}_0, \quad w_0 = w_0' = w_0'' = 0. \tag{A 7}$$

The result obtained makes it possible to formulate the problem at level $z = 0$ in terms of horizontal component velocity $\boldsymbol{v}_0$ in the closed form

$$\partial_t \boldsymbol{v}_0 + (\boldsymbol{v}_0 \cdot \nabla)\,\boldsymbol{v}_0 = -\nabla\left(\frac{p_0}{\rho}\right) + \nu\nabla^2\boldsymbol{v}_0 - \frac{2\nu}{d^2}\boldsymbol{v}_0, \quad \nabla \cdot \boldsymbol{v}_0 = 0. \tag{A 8}$$

where p_0 is the pressure distribution on the free water surface.

Appendix B. Evaluations of integrals

Before proceeding to the estimation of interaction factors e_{kn}, e_k, γ_k and area s_k of the vortex cells, it is convenient to take corresponding integrals (23), (29), (30) and (32) in the W-coordinate system. To do this, we will use representation $W = \rho \exp \mathrm{i}\varphi$ in polar coordinates ρ and φ, and also expression $\mathrm{d}\boldsymbol{x} = |\mathrm{d}z/\mathrm{d}W|^2 \rho \mathrm{d}\rho \mathrm{d}\varphi$.

B.1. *Evaluation of* $e_{k,n}$

Interaction factors $e_{k,n}$ can be reduced to the form

$$e_{k,n} = \frac{2-\delta_{kn}}{c^4} \int \theta_k \theta_n' \left[\frac{\left(\rho^4 + a^4 - 2\rho^2 a^2 \cos 2\varphi\right)\left(\rho'^4 + a^4 - 2\rho'^2 a^2 \cos 2\varphi'\right)}{\left(\rho^4 + 1 - 2\rho^2 \cos 2\varphi\right)\left(\rho'^4 + 1 - 2\rho'^2 \cos 2\varphi'\right)}\right]^{1/2}$$

$$\times \ln\left|\frac{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi')}{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi + \varphi')}\right|^{1/2} \rho\rho' \mathrm{d}\rho \mathrm{d}\varphi \mathrm{d}\rho' \mathrm{d}\varphi', \tag{B 1}$$

where the integral in the right-hand side of (B 1) is taken over intervals $\rho_{k-1} \leqslant \rho \leqslant \rho_k$, $\rho_{n-1} \leqslant \rho' \leqslant \rho_n$ and $0 \leqslant \varphi \leqslant \pi$, $0 \leqslant \varphi' \leqslant \pi$.

B.2. *Evaluation of* e_k, γ_k, s_k

After transforming into the polar W-coordinate system, interaction factors e_k, γ_k and areas s_k are reduced to

$$e_k = \frac{1}{c^2}\int_{\rho_{k-1}}^{\rho_k}\int_0^\pi \left[\frac{\left(\rho^4 + a^4 - 2\rho^2 a^2 \cos 2\varphi\right)}{\left(\rho^4 + 1 - 2\rho^2 \cos 2\varphi\right)}\right]^{1/2} \rho^2 \sin(\varphi)\, \mathrm{d}\rho \mathrm{d}\varphi, \tag{B 2}$$

$$\gamma_k = \frac{1}{c^2}\int_{\rho_{k-1}}^{\rho_k}\int_0^\pi \left[\frac{\left(\rho^2 + a^2 + 2\rho a \cos\varphi\right)}{\left(\rho^2 + a^2 - 2\rho a \cos\varphi\right)\left(\rho^4 + 1 - 2\rho^2 \cos 2\varphi\right)}\right]^{1/2} \rho^2 \sin(\varphi)\, \mathrm{d}\rho \mathrm{d}\varphi, \tag{B 3}$$

$$s_k = \frac{1}{c^2}\int_{\rho_{k-1}}^{\rho_k}\int_0^\pi \left[\frac{\left(\rho^4 + a^4 - 2\rho^2 a^2 \cos 2\varphi\right)}{\left(\rho^4 + 1 - 2\rho^2 \cos 2\varphi\right)}\right]^{1/2} \rho \mathrm{d}\rho \mathrm{d}\varphi. \tag{B 4}$$

For integrals (B 2)–(B 4) we obtain the following leading-order approximation:

$$e_N = \frac{1}{c^2}\int_{\rho_{N-1}}^{a}\int_0^\pi \left(\rho^4 + a^4 - 2\rho^2 a^2 \cos 2\varphi\right)^{1/2} \sin(\varphi)\, \mathrm{d}\rho \mathrm{d}\varphi \simeq 2\frac{a^2}{c^2}\left(C_e a - \rho_{N-1}\right), \tag{B 5}$$

	$k = N$	$N > k > 1$	$k = 1$
e_k	$2\dfrac{a^2}{c^2}(C_e a - \rho_{N-1})$	$2\dfrac{a^2}{c^2}(\rho_k - \rho_{k-1})$	$2\dfrac{a^2}{c^2}(\rho_1 - 1)$
γ_k	$\dfrac{2}{c^2}\left(C_\gamma a - \rho_{N-1}\right)$	$\dfrac{2}{c^2}(\rho_k - \rho_{k-1})$	$\dfrac{2}{c^2}(\rho_1 - 1)$
s_k	$\pi\dfrac{a^2}{c^2}\left(\ln\left(\dfrac{a}{\rho_{N-1}}\right) + C_s\right)$	$\pi\dfrac{a^2}{c^2}\ln\left(\dfrac{\rho_k}{\rho_{k-1}}\right)$	$\pi\dfrac{a^2}{c^2}\ln(2\rho_1)$

TABLE 1. Coefficients of vortex cell interaction with longshore current e_k, factors γ_k and areas of cells s_k for a rectangular basin

$$\gamma_N = \frac{1}{c^2}\int_{\rho_{N-1}}^{a}\int_0^{\pi}\left[\frac{(\rho^2 + a^2 + 2\rho a\cos\varphi)}{(\rho^2 + a^2 - 2\rho a\cos\varphi)}\right]^{1/2}\sin(\varphi)\,\mathrm{d}\rho\mathrm{d}\varphi \simeq \frac{2}{c^2}\left(C_\gamma a - \rho_{N-1}\right), \quad \text{(B 6)}$$

$$s_N = \frac{1}{c^2}\int_{\rho_{N-1}}^{a}\int_0^{\pi}(\rho^4 + a^4 - 2\rho^2 a^2\cos 2\varphi)^{1/2}\rho^{-1}\mathrm{d}\rho\mathrm{d}\varphi \simeq \pi\frac{a^2}{c^2}\left(\ln\left(\frac{a}{\rho_{N-1}}\right) + C_s\right). \quad \text{(B 7)}$$

Here $C_e = 1.16$, $C_s = 1.70$ are the correction factors, the numerical values of which are given by integrals

$$\left.\begin{aligned} C_e &= \frac{1}{2}\int_0^1 t\left[\frac{1-t^2}{t} + \left(\frac{1+t^2}{t}\right)^2\arctan(t)\right]\mathrm{d}t = \tfrac{1}{2}\mathrm{G} + \frac{\pi}{4} - \frac{1}{12}, \\ C_s &= \int_0^1\left[t + \frac{1+t^2}{t}\arctan(t)\right]\mathrm{d}t = \mathrm{G} + \frac{\pi}{4}, \end{aligned}\right\} \quad \text{(B 8)}$$

where $\mathrm{G} = 0.916$ is Catalan's constant. The choice of correction factor C_γ is made in agreement with the obvious condition, $\sum_{k=1}^{N} s_k = hl$, from which it follows that $C_\gamma = \ln 2 - 1 \simeq -0.31$. The leading-order approximations for e_k, γ_k, s_k are given in table 1.

REFERENCES

ARNOLD, V. I. 1969 The Hamiltonian nature of the Euler equations in the dynamics of a rigid body and of a perfect fluid. *Russ. Math. Surveys* **24** (3), 225–226.

BATCHELOR, G. K. 1956 On steady laminar flow with closed stream lines at the large Reynolds number. *J. Fluid Mech.* **1**, 177–190.

BISKAMP, D. & WELTER, H. 1990 Magnetic field amplification and saturation in two-dimensional magnetohydrodynamic turbulence. *Phys. Fluids* B **2**, 1787–1793.

CARNEVALE, G. F. & VALLIS, G. K. 1990 Pseudo-advective relaxation to stable states of inviscid two-dimensional fluids. *J. Fluid Mech.* **213**, 549–571.

CHERNYSHENKO, S. I. 1988 An asymptotical behaviour of the steady separated flow past a body at the large Reynolds numbers. *Prikl. Mat. i Meh.* **52**, 958–966.

CHERNYSHENKO, S. I. & CASTRO, I. P. 1993 High-Reynolds-number asymptotics of the steady flow through a row of bluff bodies. *J. Fluid Mech.* **257**, 421–449.

COUDER, Y. & BASDEVANT, C. 1986 Experimental and numerical study of vortex couples in two-dimensional flows. *J. Fluid Mech.* **173**, 225–251.

DRITSCHEL, D. G. 1993 Vortex properties of two-dimensional turbulence. *Phys. Fluids* A **5**, 984–997.

Gledzer, E. B., Dolzhanskii, F. V. & Obuhov, A. M. 1981 *Systems of Hydrodynamic Type and Their Application*. Nauka, Moscow.

Goncharov, V. P. & Makarov, A. L. 1996 Vortex structures in quasi-conservative flow with additional motion integral due to viscosity. *Sov. Phys. Dokl.* **346**, 1–4.

Goncharov, V. P. & Pavlov, V. I. 1993 *Problems of Hydrodynamics in Hamiltonian description*. Moscow University Press.

Goncharov, V. P. & Pavlov, V. I. 1997 Some remarks on physical fondation of the Hamiltonian description of fluid motions. *Eur. J. Mech.* B *Fluids* **16**, 509–555.

Happel, J. & Brenner, H. 1965 *Low Reynolds Number Hydrodynamics*. Prentice-Hall.

Holm, D. D., Marsden, J. E., Ratin, T. & Weinstein, A. 1985 Nonlinear stability of fluid and plasma equilibria. *Phys. Rep.* **123**, 1–116.

Lavrent'ev, M. A. & Shabat, B. V. 1965*a* *Methods of Theory of Complex Variable Functions*. Nauka, Moscow.

Lavrent'ev, M. A. & Shabat, B. V. 1973*b* *Hydrodynamic problems and their mathematical models*. Nauka, Moscow.

Mills, R. D. 1965 On the closed motion of a fluid in a square cavity. *J. R. Aero. Soc.* **69,** 116–120.

Montgomery, D., Matthaeus, W. H., Stribling, W. T., Martinez, D. & Oughton, S. 1992 Relaxation in two dimensions and the "sinh-Poisson" equation. *Phys. Fluids* A **4**, 3–6.

Nguyen Duc, J.-M. & Sommeria, J. 1988 Experimental characterization of steady two-dimensional vortex couples. *J. Fluid Mech.* **192**, 175–192.

Olver, P. J. 1986 *Applications of Lie Groups to Differential Equations*. Springer.

Pan, F. & Acrivos, A. 1967 Steady flows in rectangular cavities. *J. Fluid Mech.* **28**, 643–655.

Pasmanter, R. A. 1994 On long-lived vortices in 2D viscous flows, most probable states of inviscid 2-D flows and a soliton equation. *Phys. Fluids.* **6**, 1236–1241.

Petviashvili, V. I., Pokhotelov, O. A. & Chudin, N. V. 1982 Solitary toroidal vortices. *Sov. Phys. JETP*. **82**, no. 6, 1833–1839.

Prandtl, P. 1905 Uber Flussikeitsbewegung beu sehr kleiner Reibung. *Vhr. d. III Intern. Math.-Kongr. Heiderlberg, 1904, Leipzig: Tebubner*. 484–491.

Robert, R. & Sommeria, J. 1991 Statistical equilibrium states for two-dimensional flows. *J. Fluid Mech.* **229**, 291–310.

Saffman, P. G. 1981 Dynamics of vorticity. *J. Fluid Mech.* **106**, 49–58.

Salmon, R. 1988 Hamiltonian fluid mechanics. *Ann. Rev. Fluid Mech.* **20**, 225–256.

Schlichting, H. 1955 *Boundary Layer Theory*. McGraw-Hill.

Shen, H. 1992 Invariant two-dimensional structures of vorticity field in nonviscous fluid. *Phys. Fluids* A **4**, 2027–2035.

Shepherd, T. G. 1990 A general method for finding extremal states of Hamiltonian dynamical systems, with applications to perfect fluids. *J. Fluid Mech.* **213**, 573–587.

Vallis, G. K., Carnevale, G. F. & Young, W. R. 1989 Extremal energy properties and construction of stable solutions of the Euler equations. *J. Fluid Mech.* **196**, 291–322.

Weiss, R. F. & Florsheim, B. H. 1965 Flow in a cavity at low Reynolds number. *Phys. Fluids* **8**, 1631–1635.